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When these 6×6 squares are made pan-diagonal, i. e., perfect in all their diagonals, the normal couplets are arranged in harmonic relation throughout the square, the two paired numbers that equal n^2+1 being always located in the same diagonal and equally spaced $n/2$ cells apart. If the square is made strictly magic, however, this harmonic arrangement of the couplets is naturally disturbed in the imperfect diagonals.

The above remarks and rules will of course apply generally to 10×10 and larger squares of this class. A 10×10 square modified from Mr. Savage's example to secure the harmonic arrangement of the couplets, as above referred to, is given in Fig. 8.

20	41	86	32	73	75	31	85	43	19
89	36	79	1	48	46	2	80	34	90
63	5	50	93	40	39	95	49	7	64
56	97	23	68	9	10	66	24	98	54
25	72	14	59	84	83	57	13	71	27
26	70	16	58	82	81	60	15	69	28
55	99	21	67	11	12	65	22	100	53
62	6	52	94	37	38	96	51	8	61
91	35	77	3	47	45	4	78	33	92
18	44	88	30	74	76	29	87	42	17

+1 —1

Fig. 8.

W. S. Andrews.

Schenectady, N. Y.

BOLYAI, LOBATCHEVSKY, RUSSELL.

In his book on the *Foundations of Mathematics,* Dr. Carus wonders what sort of pistols Bolyai used in the famous duels where, like Ivanhoe, he discomfited the whole series of challengers. But Bolyai was a compound of Saladin and Richard, fighting with a Damascus blade which cut silken cushions or chopped iron. Franz Schmidt told me in Budapesth that his father had seen Bolyai lop off a spike driven into his doorpost, and that some of his duels were to the death.

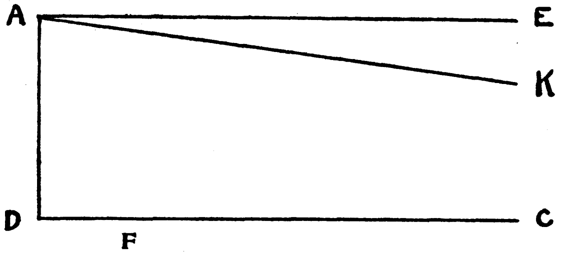
Bolyai published only 24 pages of text, but this is the most extraordinary two dozen pages in the history of human thought. The very first page would kill or cure my genial friend Mr. Russell.

I fear the outcome, since on page 625 of *The Monist,* Vol. XIX, he says: "As to such an idea as that two lines may be parallel if they are taken in the same *sense,* and yet *not* parallel if taken in opposite senses, I fail to find any vestige of it in Lobatchevsky's text. That would be to make Lobatchevsky's system a system of vectors instead of a geometry, and I am sure such a system as well as the idea of a *sensed relation* would put me to permanent intellectual confusion."

The confusion is already here, since vectors assume translation independent of rotation, that is assume the parallel postulate, the whole question at issue.

The Bolyai-Lobatchevsky geometry, then, is of course not a system of vectors, but it is largely a system of *rays.* Bolyai says the ray AB is that half of the straight AB which commences at the point A and contains the point B. Then § 1, "If the ray DC is not cut by the ray AE, situated in the same plane, but is cut by every ray AK comprised in the angle DAE, this is designated by AE||| DC.

It is evident that *there is one such ray* AE, *and only one,* from any point A outside the straight DC, and that the sum of the angles CDA, DAE does not exceed two right angles.



It is also clear that AE|||FC, wherever the point F be taken on the straight DC, supposing in all such cases DC>DF; that is, ray FC of the same sense as ray DC.

Bolyai translates by the word *asymptote* his symbol |||.

To a given ray DC, from a given point A outside it, there is one and only one asymptote AE. Bolyai calls parallel to the straight DC every coplanar straight through A which nowhere cuts the straight DC. So through A, in his terminology, there is only one asymptote to the ray DC, but there may be an infinity of parallels to the straight DC.

Philip Kelland, senior wrangler and tutor to Sylvester, whom he called the greatest mathematician in the world, was an independent discoverer of the non-Euclidean geometry, and like Bolyai, he called parallel to a given straight all straights coplanar with it which nowhere met it. But this meaning and use of the word parallel has been superseded by Lobatchevsky's. He has no name for this possible infinity of straights through A coplanar with the straight DC but nowhere cutting it. But that one of them which contains the Bolyai asymptote to the ray DC, Lobatchevsky calls parallel to DC and then says "upon the other side of the perpendicular from A will lie also a line AM, parallel to the prolongation DC′ of DC, so that under this assumption we must also make a distinction of *sides in parallelism.*"

This *sensed relation* then is Lobatchevsky's parallelism, and that it is a sensed relation, a one-sided relation, a one-way relation, a relation which goes toward one side only is stressed and emphasized by his § 24: "The farther parallel lines are prolonged *on the side of their parallelism,* the more they approach one another." In other words, Lobatchevsky parallels are one-way asymptotes.

So his next theorem, § 25, which tripped Mr. Russell, I might have translated: Two straight lines which are same-way asymptotes to a third are also asymptotes to one another.

So of all straights coplanar with a given horizontal straight which nowhere cross it, two through each given point are parallel to it, one to the right, the other to the left, and all the others I call *ultra-parallel.*

A parallel to a straight meets it at infinity. An ultra-parallel does not even meet it at infinity. Parallels are straights with a common point at infinity. Parallels are straights which meet on a figurative point. If ultra-parallels determine a point, we must have another name for it. Call it an ideal point. Thus equipped, we are able to answer Mr. Russell's question, *Monist,* page 621: "How is the professional expert (the man who knows non-Euclidean geometry) better fitted to see more lucidly in dealing with the elements of geometry than any other person of good geometric faculty?"

Just thus, my dear friend: You say, "I will now spread before the reader in detail what seems to me to be good geometrical proof of my proposition. Consider and refer to the following figure." I do, and instantly, at a glance, I see your fallacy, your *petitio principii.* I am saved the reading of your three pages of pseudo-proof. You assume that an angle is determined in size because it is made by the side with the hypotenuse of an isosceles right-angled triangle. You say, p. 624: "Now the angle MEN being a *w*-angle equals the angle $\beta\epsilon\delta$..."; in other words, you assume that the size of the isosceles right triangle has no effect on the size of its acute angles. This is Wallis's form of the parallel postulate. And so you are guilty of begging the question.

Not only would a smattering of Bolyai have saved you but so would a little excursion into Chapter XV of my *Rational Geometry,* which starts by saying: "Deducing spherics from a set of assumptions which give no parallels, *no similar figures,* we get a two-dimensional non-Euclidean geometry, yet one whose results are also part of three-dimensional Euclidean."

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A REMARK ON F. C. RUSSELL'S THEOREM.

F. C. Russell of Chicago has endeavored in the April number of *The Monist* to disprove the legitimacy of the non-Euclidean geometry by showing the demonstrability of the parallel postulate. The basis of his considerations has been laid on a simple proposition that the angle-sum of an isosceles right-angled triangle equals two right angles; a proposition the proof of which he does not dare give, saying it would be "spreading an imputation upon the reader," being so simple in nature. But the whole secret of the matter remained concealed under this unknown sort of a proof, and so we are lucky that we had it imparted to us by Russell himself in a subsequent number. In studying it, we have found all that can be desired.

Russell defines his *u*-angles "as being such angles as the sides of an isosceles right-angled triangle make with the hypotenuse." This definition is of course not in any way objectionable, but when Russell has to consider the *u*-angles arising from different triangles of unequal sizes, to be always equal, he has unconsciously fallen into a pit of thought, from which he is unable to get out. When we adhere to the Euclidean world, we can well prove the assumption Russell makes, but how can he protest the legitimacy of it, when he is going to show the Euclidean system to be the sole one that can be relied upon? If he wants to be credited by us, he must first prove the assumption he has made; which most probably he cannot do without having recourse to the parallel postulate or some-